# Magic Squares in Linear Algebra

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April 19, 2021

**Abstract**

This document illustrates much of the structure behind what it means to be a magic square in linear algebra by reiterating on Christopher J. Henrich's work with the intention of enlightening an avid linear algebra student or software engineer. Inside you'll find answers on what it means to be a magic square, how to detect one, understanding their structure, how many may exist, as well as the possibility of other exotic patterns.

A magic square is a matrix with distinct entries in which every row, column, and diagonal share a common sum. A more detailed definition is still to follow. The following is but one example of a magic square of order 4 with a sum of 34 known as Dürer's Square.

| | | | |
|---|---|---|---|
| 16 | 3 | 2 | 13 |
| 5 | 10 | 11 | 8 |
| 9 | 6 | 7 | 12 |
| 4 | 15 | 14 | 1 |

# Contents

# 1 Introduction

A magic square of order n is an arrangement of the numbers

$$\{1, 2, ..., n^2\}$$

as an n by n matrix for some positive integer n such that each row, column, and diagonal family add up to the same sum S.

$$1 + 2 + ... + n^2 = nS \qquad \leftrightarrow \qquad S = \frac{n(n^2+1)}{2}$$

To help clarify our understanding, lets define the class of diagonal families. The most familiar may be that of the main-diagonal and off-diagonal as transcribed in Figure 1. Magic squares that satisfy this property are often known as Ordinary. The next set of diagonals is that of the quasi-pandiagonal which can be observed in Figure 2. We will also be looking at other parallel families of this as seen in Figure 3. Magic squares with magic properties over these sets are known as Panmagic Squares and will be the highlight of this paper.

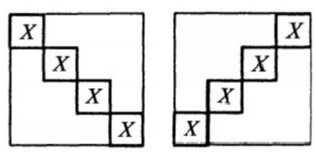



Figure 1: Main and off diagonals.

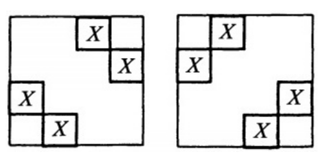



Figure 2: Quasi-pandiagonal.

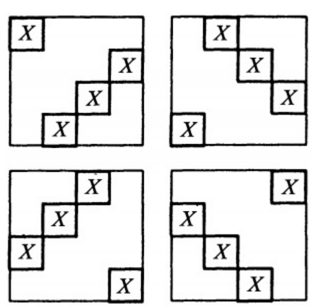
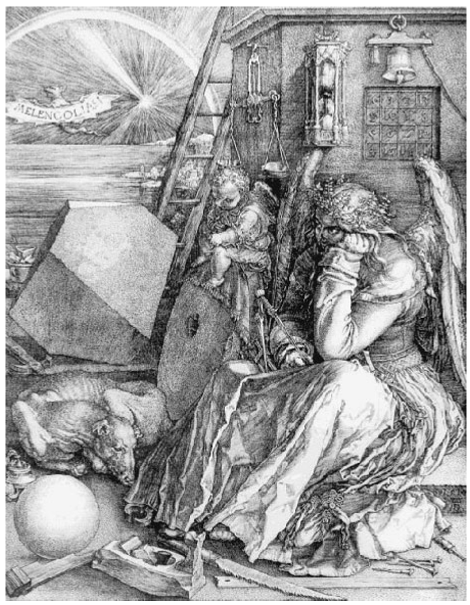



Figure 3: Broken diagonal.

## 2 Dürer's Square

First we will look at the structure of the Dürer Square, named after Albrecht Dürer (1471-1528) who designed the Melancholia; the earliest record of an order 4 panmagic square as seen in the background on the top right of the Melancholia. We start by developing a labeling convention for the entries of a magic square and a method of describing subspaces.



Albrecht Dürer's "Melancholia"

| | | | |
|---|---|---|---|
| 16 | 3 | 2 | 13 |
| 5 | 10 | 11 | 8 |
| 9 | 6 | 7 | 12 |
| 4 | 15 | 14 | 1 |

Dürer's Square

## 2.1 Denoting Subspaces

Consider the four by four matrix with numbered rows and columns in base-2 as seen in Figure 4. Since each entry of this matrix can be coordinated to uniquely, we can number the entries one through sixteen by taking the row number in which they appeared in concatenated by which column they appeared in, resulting in entries from the base-2 integer field of length four, denoted ${F_2}^4$.

| | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| 00 | 0000 | 0001 | 0010 | 0011 |
| 01 | 0100 | 0101 | 0110 | 0111 |
| 10 | 1000 | 1001 | 1010 | 1011 |
| 11 | 1100 | 1101 | 1110 | 1111 |

Figure 4: Basis of ${F_2}^4$.

**Definition of a subspace:** A subset U is a subspace of a finite-dimensional vector space V if and only if U contains the additive identity, is closed under vector addiction, and scalar multiplication.

We focus our attention to subspaces of ${F_2}^4$ as they give us a handle on describing groups of entries of our Dürer's Square. We start by defining the natural basis of ${F_2}^4$ as the list $(e_1, e_2, e_3, e_4) = (0001, 0010, 0100, 1000)$ which can be thought of as the digits of our entries written in base-2.

Referring to figure 4, we can see that the $1^{st}$ row is spanned by the basis vectors $(e_3, e_4)$ since the only spanning digits are in the $3^{rd}$ and $4^{th}$ placeholder. Also, we can view the $1^{st}$ column subspace by $(e_1, e_2)$. Next, the main diagonal may be spanned by $(e_1 + e_3, e_2 + e_4)$. Of the broken diagonals we can describe the $1^{st}$ broken diagonal in figure 3 by $(e_1 + e_2 + e_4, e_2 + e_3 + e_4)$. Now that we have a way of describing each of the basic subspace, we can begin to find the remaining subspaces that makeup a Dürer Square by observing the affine parallel subspaces.

**Definition of an affine (parallel) subset:** An affine subset of V is a subset of V of the form v + U for some vector v in V and subspace U of V.

To get a better grasp with working on affine subspaces, one possible example may be that of describing the last row of the Dürer Square by looking at Figure 4. We start by taking the subspace that represents the $1^{st}$ row and add an appropriate element of ${F_2}^4$, namely $1100_2$, resulting in the following basis.

$$(e_3, e_4) + 1100_2 = (e_3 + 0011_2, e_4 + 1100_2)$$

Another example may be the representation of the last column of the Dürer Square. We start by taking the subspace representation for the $1^{st}$ column and add the appropriate element from ${F_2}^4$, namely $1100_2$.

$$(e_1, e_2) + 0011_2 = (e_1 + 0011_2, e_2 + 0011_2)$$

Figuring out which element of ${F_2}^4$ we need for describing a specific row or column may be intuitive as we just observe which digits aren't spanned by our basis vectors. This however still raises the question about describing the diagonal families. We have already previously described the main-diagonal family, but we can just simply add another appropriate element from ${F_2}^4$, namely $1100_2$.

$$(e_1 + e_3, e_2 + e_4) + 1100_2 = (e_1 + e_3 + 1100_2, e_2 + e_4 + 1100_2)$$

This however still doesn't satisfy our understanding of the other diagonal families. We can describe the $1^{st}$ quasi-pandiagonal that appears in Figure 2 by adding a different element from ${F_2}^4$ to our main diagonal, namely 0111 (and similarly, the $2^{nd}$ quasi-pandiagonal can be described by adding 1110).

$$(e_1 + e_3, e_2 + e_4) + 0111_2 = (e_1 + e_3 + 0111_2, e_2 + e_4 + 0111_2)$$

And lastly, we need a way to describe our final diagonal family. Starting with our description for the $1^{st}$ broken diagonal in Figure 3, we can represent the 2nd family by adding the appropriate element from ${F_2}^4$, $1100_2$ (or the 3rd and 4th families by adding $0010_2$ and $1111_2$ respectively. It becomes apparent that the diagonal families of Figure 2 and Figure 3 share much in common, being that they are affine parallel to one another.

$$(e_1 + e_3, e_2 + e_4) + 1100_2 = (e_1 + e_3 + 1100_2, e_2 + e_4 + 1100_2)$$

## 2.2 Denoting Linear Functions

Next we begin by taking the Dürer Square square in $F_2{}^4$ as shown in Figure 5 (after subtracting 1 from each entry for ease of representation) and consider the parity matrix of the Dürer Square as shown in Figure 6. We can view this matrix as a linear function f from $F_2{}^4$ to ($F_2{}^4$ to $F_2$). Its worth mentioning that this new matrix preserved its panmagic property. This now brings our attention to describing a respective dual basis ($f_1$, $f_2$, $f_3$, $f_4$) for $F_2$ that corresponds to ($e_1$, $e_2$, $e_3$, $e_4$) of $F_2{}^4$.

| | | | |
|---|---|---|---|
| 1111 | 0010 | 0001 | 1100 |
| 0100 | 1001 | 1010 | 0111 |
| 1000 | 0101 | 0110 | 1011 |
| 0011 | 1110 | 1101 | 0000 |

Figure 5: Dürer's square in $F_2{}^4$.

| | | | |
|---|---|---|---|
| 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 |

Figure 6: Dürer's square parity matrix.

**Definition of Dual Basis:** If ($v_1$, ..., $v_n$) is a basis of V then its dual basis is ($w_1$, ..., $w_n$) of V' (the vector space of all linear functionals on V) where each w is a linear functional on V such that $w_i(v_j) = \{1$ if $i = j$, or 0 if $i \neq j\}$.

Referring back to the parity matrix of Dürer's Square (Figure 6), we can describe this matrix in terms of linear functions by the following equation.

$$f_1 + f_2 + f_4$$

To confirm this, we use our matrix from Figure 4 to see what exactly this means. Applying each of $f_1$, $f_2$, and $f_4$ to the natural basis of $F_2{}^4$, ($e_1$, $e_2$, $e_3$, $e_4$), yields the following 3 matrices respectively as seen in Figure 7. Taking the sum of these 3 matrices yields our parity Dürer matrix (Figure 6).

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 |

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 |
| 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 |

| | | | |
|---|---|---|---|
| 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 1 |

Figure 7: Linear functions $f_1$, $f_2$, and $f_4$.

Since each of these linear functions are either constant, or more importantly, take on the values 0 and 1 equally often on a vector subspace, then it will be true for any affine parallel subspace. This brings us to the following proposition about Figure 6 having the panmagic property that it does.

**Proposition 1:**

Let $\psi$ be a linear function on $F_2{}^4$ with values in $F_2$; let E be a linear subspace of $F_2{}^4$ which is not contained in the null-space of $\psi$. Then, on any affine subspace parallel to E, the function $\psi$ takes the values 0 and 1 equally often.

Now lets see if we can quantify the entirety of the magic properties that are in the Dürer Square's square by referring back to the binary representation as seen in Figure 5 and viewing each digit separately. Let $V_i$ represent the Dürer Square's $i^{th}$ digits in $F_2$. We can disect the Dürer Square into the four squares as seen in Figure 8. These functions of $V_i$ are really just affine linear functions amongst each other as written in Figure 9.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 1 0 0 1 | | 1 0 0 1 | | 1 1 0 0 | | 1 0 1 0 |
| $V_1$: | 0 1 1 0 | $V_2$: | 1 0 0 1 | $V_3$: | 0 0 1 1 | $V_4$: | 0 1 0 1 |
| | 1 0 0 1 | | 0 1 1 0 | | 0 0 1 1 | | 0 1 0 1 |
| | 0 1 1 0 | | 0 1 1 0 | | 1 1 0 0 | | 1 0 1 0 |

Figure 8: Digits of Dürer's square.

$$\begin{aligned} V_1 &= \quad\;\; f_2 + f_3 + f_4 + 1 \\ V_2 &= f_1 \quad\;\; + f_3 + f_4 + 1 \\ V_3 &= f_1 + f_2 + f_3 \quad\;\; + 1 \\ V_4 &= f_1 + f_2 \quad\;\; + f_4 + 1 \end{aligned}$$

Figure 9: Associated linear functions.

Each value function V is non-constant on the subspaces corresponding to each of our row, column, or diagonal family of interest as well as their affine parallel subspaces. So a non-constant affine function W must take the values 0 and 1 equally often. We can describe this function W as:

$$w(x) = 8V_1 + 4V_2 + 2V_3 + V_4 + 1$$

# 3 Generalizing Dürer Square

Much of this section was derived from Christopher J. Henrich's work and plays a core part in generalizing the Dürer Square. Define V to be the set $V_i$, $i = 1, ..., 4$, of affine functions on $F_2{}^4$ with values in $F_2$. Denote the linear part of $V_i$ by $\psi_i$ when discussing a particular affine square W.

**Definition 1:** Given V, then the map W from $F_2{}^4$ to Z is an affine square.

**Definition 2:** Given V, and letting W be the affine square determined by them; let E be a linear subspace of $F_2{}^4$. E is magic for W if each of the linear functions $\psi_i$ is non-zero on E.

**Proposition 2:** Given V, and letting E be a linear subspace of $F_2{}^4$ which is magic for the affine square W determined by $V_i$, then W has uniform sums on the affine subspaces parallel to E.

Proof: Let E have dimension d. By Proposition 1, each of the functions $\psi_i$ takes the values 0 and 1 each $2^{d-1}$ times on every affine space parallel to E (as well as $V_i$). Therefore each sum of $V_i$ over every affine subspace is $2^{d-1}$; hence equation W.

**Definition 3:** Given V, and letting W be the affine square determined by $V_i$, then W is nonsingular if $\psi_1, ..., \psi_4$ are linearly independent.

**Proposition 3:** Given V, and suppose that the affine square W determined by $V_i$ is nonsingular, then the numbers in W are exactly 1, ..., 16.

Proof: Because $1 \leq W(x) \leq 16$ for $x \in F_2{}^4$, and there are 16 points in $F_2{}^4$, it is sufficient to show that no two of them have the same value of W. Now suppose x and y are members of $F_2{}^4$ and $V_i(x) = V_i(y)$ for $i = 1, ..., 4$. Then we have $\psi_i(x) = \psi_i(y)$ for all i, but by hypothesis the $\psi_i$ are linearly independent, and therefore span $F_2{}^{4*}$. Thus $x = y$.

**Definition 4:** An affine magic square is a nonsingular affine square for which subspaces $(e_1, e_2)$, $(e_3, e_4)$, and $(e_1 + e_2, e_3 + e_4)$ are magic.

**Proposition 4:** Every affine magic square is a pandiagonal or quasi-pandiagonal magic square.

Proof: This follows from propositions 2 and 3.

In summary, we correlate $F_2{}^4$ with Z through an affine nonsingular square W which holds our magic property so long as each linear function $\psi_i$ on $F_2{}^4$ determined by W is nonzero. Each of our 16 entries will appear distinctly.

# 4 How Many Magic Squares Exist?

We will be paying close attention to each linear function $f_i$ on $F_2{}^4$. For $f_i$ to be non-constant on a particular subspace then we must restrict this linear function to maintain our magic property. In section 2.1: Denoting Subspaces, we noticed that the $1^{st}$ row is spanned by the basis vectors ($e_3$, $e_4$) and that the other rows were just affine parallel to this one. So we must restrict our subspace to being nonzero over $e_3$ or $e_4$. Similarly, the column subspace must be nonzero over ($e_1$, $e_2$). Furthermore, we saw that each of the diagonal families were simply affine parallel to one another, so we need only restrict our diagonal subspace to being nonzero over ($e_1 + e_3$, $e_2 + e_4$).

This gives us a base on generating what turns out to be the six linear functions that satisfy our constraints as seen in Figure 10. Some of these linear expressions shouldn't come as a surprise as we have seen this type of behavior previously in Figure 9 in determining each $V_i$ which represented each digit in our binary representation of Dürer's square. We observed that even the digits held this magic property that we seek. It turns out that there are still two more linear functions, $L_1$ and $L_5$, which preserve this magic property that wasn't mentioned previously. For visual completeness, Figure 11 includes the remaining two value functions $V_5$ and $V_6$ respective to these two linear functions.

$$L_1 = f_2 + f_3; \quad L_2 = f_1 + f_3 + f_4; \quad L_3 = f_1 + f_2 + f_4;$$
$$L_4 = f_2 + f_3 + f_4; \quad L_5 = f_1 + f_4; \quad L_6 = f_1 + f_2 + f_3.$$

Figure 10: Relation candidates.

$$V_5 = \begin{matrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{matrix} \qquad V_6 = \begin{matrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{matrix}$$

Figure 11: More associated linear functions.

In determining an affine magic square, we need to select four functions ($f_1$, $f_2$, $f_3$, and $f_4$) such that they hold no linear relationship to preserve our magic property. A linear relationship may result in the possibility of falsely determining a magic square who's row, column, or diagonal subspace turns out to be contained in the null space of our linear relations and hence be zero across all entries. We find that two such relations amongst those proposed in Figure 10 arise as listed in Figure 12.

$$L_1 + L_2 + L_3 = 0$$
$$L_4 + L_5 + L_6 = 0$$

Figure 12: Satisfying relations.

We can express a relation among three functions as $L_a + L_b + L_c = 0$ which can only be zero in one way. However when trying to express four functions $L_a + L_b + L_c + L_d = 0$, we find that a nonzero function that isn't eligible can be expressed as a sum of two eligible functions hence no four-term relations exist among these eligible functions.

Given the 6 eligible functions from Figure 10 we can choose 4 in 15 ways. 2 of the 3-term relations exclude 3 sets so only 9 linearly independent sets of eligible linear functions remain.

Since each linear function takes on the values 0 and 1 equally often, and we are looking at cardinality-4 sets of these affine functions, then there are 16 sets. Since we can arrange these 4 affine functions in 4! (24) ways, and that there are 8 geometric symmetries of a square, then the total number of affine magic squares of order 4 is:

$$9 \cdot 16 \cdot 24/8 = 432$$

This means that there are 432 algebraic magic squares of order 4. A complete enumeration of magic squares of order 4 was discovered in the seventeenth century by Bernard Frénicle de Bessey. Also, Bensen and Jacoby confirmed this computationally, as well as Ollerenshaw and Bondi more analytically. These studies suggests that the number of affine magic squares coincide with the number of algebraic magic squares. This gives us the final proposition as follows.

**Proposition 5:** Every pandiagonal or quasi-pandiagonal magic square of order 4 is an affine magic square.

We can visualize these order 4 magic square patterns by taking a number and pairing it with its counterpart by a line such that their sum is 17 as seen in Figure 13.

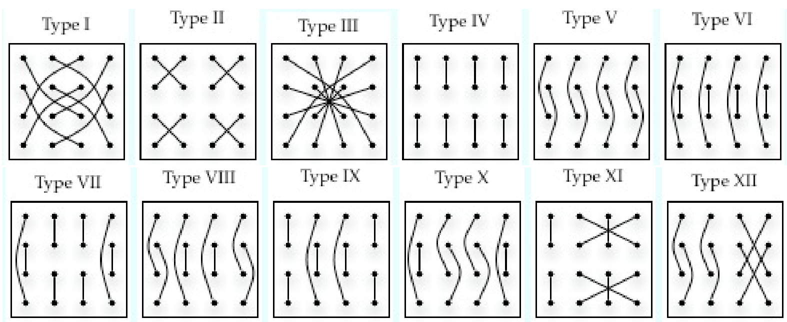



Figure 13: Types of order 4 magic squares.

If L is the linear part of the square mapping, and q is a vector of $F_2^4$ such that Lq = (1,1,1,1), then two cells in the square have complementary values if and only if they're the images under the position map of vectors which differ by q. Each complementary pair is the image of a line parallel to (0,q). The equation Lq = (1,1,1,1) depends on the set of eligible linear functions (and not their order). The 9 independent sets of eligible functions that determine q are seen in Figure 14 where the last column represents which type of pattern as we saw in Figure 13.

| Linear parts | $q$ | Class |
|---|---|---|
| $L_1$ $L_2$ $L_4$ $L_5$ | (1,1,0,0) | VI |
| $L_1$ $L_2$ $L_4$ $L_6$ | (0,0,1,0) | V |
| $L_1$ $L_2$ $L_5$ $L_6$ | (0,1,0,1) | II |
| $L_1$ $L_3$ $L_4$ $L_5$ | (1,0,1,0) | I |
| $L_1$ $L_3$ $L_4$ $L_6$ | (0,1,0,0) | IV |
| $L_1$ $L_3$ $L_5$ $L_6$ | (0,0,1,1) | VI |
| $L_2$ $L_3$ $L_4$ $L_5$ | (0,0,0,1) | IV |
| $L_2$ $L_3$ $L_4$ $L_6$ | (1,1,1,1) | III |
| $L_2$ $L_3$ $L_5$ $L_6$ | (1,0,0,0) | V |

Figure 14: Linear relations and their square types.

The number of distinct squares generated by each of these patterns is:

$$24 \cdot 168 = 48$$

Reflecting about the square's diagonal corresponds to switching the $1^{st}$ and $2^{nd}$ halves of the position vector. So q = (1,0,1,0), (0,1,0,1), and (1,1,1,1) are unchanged and so their pattern is parallel, visualized by types 1, 2, and 3 respectively. The other possible values of q are interchanged so each type is contributed to by two values, and the number of quasi-pandiagonal squares visualized in types 4, 5, and 6 are:

$$(2 \cdot 24 \cdot 16)/8 = 96$$

Some of type 6, as well as the remaining types aren't affine because their complementary pairs aren't all parallel in $F_2{}^4$ in which there are two families of four parallel lines.

## 5 Benjamin Franklin's Examples

We may now begin to generalize our understanding to magic squares whose order is a power of 2. First constructed by Benjamin Franklin were the following magic squares of order 8 (Figure 15) and order 16 (Figure 16). We will only be focusing on the magic square of order 8, but much of our procedure extends to the order 16 square along with many more unexplored subspaces of various dimensions.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 52 | 61 | 4 | 13 | 20 | 29 | 36 | 45 |
| 14 | 3 | 62 | 51 | 46 | 35 | 30 | 19 |
| 53 | 60 | 5 | 12 | 21 | 28 | 37 | 44 |
| 11 | 6 | 59 | 54 | 43 | 38 | 27 | 22 |
| 55 | 58 | 7 | 10 | 23 | 26 | 39 | 42 |
| 9 | 8 | 57 | 56 | 41 | 40 | 25 | 24 |
| 50 | 63 | 2 | 15 | 18 | 31 | 34 | 47 |
| 16 | 1 | 64 | 49 | 48 | 33 | 32 | 17 |

Figure 15: Order 8 magic square.

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 200 | 217 | 232 | 249 | 8 | 25 | 40 | 57 | 72 | 89 | 104 | 121 | 136 | 153 | 168 | 185 |
| 58 | 39 | 26 | 7 | 250 | 31 | 218 | 199 | 186 | 167 | 154 | 135 | 122 | 103 | 90 | 71 |
| 198 | 219 | 230 | 251 | 6 | 27 | 38 | 59 | 70 | 91 | 102 | 123 | 134 | 155 | 166 | 187 |
| 60 | 37 | 28 | 5 | 252 | 229 | 220 | 197 | 188 | 165 | 156 | 133 | 124 | 101 | 92 | 69 |
| 201 | 216 | 233 | 248 | 9 | 24 | 41 | 56 | 73 | 88 | 105 | 120 | 137 | 152 | 169 | 184 |
| 55 | 42 | 23 | 10 | 247 | 234 | 215 | 202 | 183 | 170 | 151 | 138 | 119 | 106 | 87 | 74 |
| 203 | 214 | 235 | 246 | 11 | 22 | 43 | 54 | 75 | 86 | 107 | 118 | 139 | 150 | 171 | 182 |
| 53 | 44 | 21 | 12 | 245 | 236 | 213 | 204 | 181 | 172 | 149 | 140 | 117 | 108 | 85 | 76 |
| 205 | 212 | 237 | 244 | 13 | 20 | 45 | 52 | 77 | 84 | 109 | 116 | 141 | 148 | 173 | 180 |
| 51 | 46 | 19 | 14 | 243 | 238 | 211 | 206 | 179 | 174 | 147 | 142 | 115 | 110 | 83 | 78 |
| 207 | 210 | 239 | 242 | 15 | 18 | 47 | 50 | 79 | 82 | 111 | 114 | 143 | 146 | 175 | 178 |
| 49 | 48 | 17 | 16 | 241 | 240 | 209 | 208 | 177 | 176 | 145 | 144 | 113 | 112 | 81 | 80 |
| 196 | 221 | 228 | 253 | 4 | 29 | 36 | 61 | 68 | 93 | 100 | 125 | 132 | 157 | 164 | 189 |
| 62 | 35 | 30 | 3 | 254 | 227 | 222 | 195 | 190 | 163 | 158 | 131 | 126 | 99 | 94 | 67 |
| 194 | 223 | 226 | 255 | 2 | 31 | 34 | 63 | 66 | 95 | 98 | 127 | 130 | 159 | 162 | 191 |
| 64 | 33 | 32 | 1 | 256 | 225 | 224 | 193 | 192 | 161 | 160 | 129 | 128 | 97 | 96 | 65 |

Figure 16: Order 16 magic square.

Starting with our order 8 magic square, we begin by once again subtracting 1 from each entry and writing these numbers in base-2 so that we may view them in terms of $F_2{}^6$ as seen in Figure 17.

| | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|---|---|---|---|---|---|---|---|---|
| 000 | 110011 | 111100 | 000011 | 001100 | 010011 | 011100 | 100011 | 101100 |
| 001 | 001101 | 000010 | 111101 | 110010 | 101101 | 100010 | 011101 | 010010 |
| 010 | 110100 | 111011 | 000100 | 001011 | 010100 | 011011 | 100100 | 101011 |
| 011 | 001010 | 000101 | 111010 | 110101 | 101010 | 100101 | 011010 | 010101 |
| 100 | 110110 | 111001 | 000110 | 001001 | 010110 | 011001 | 100110 | 101001 |
| 101 | 001000 | 000111 | 111000 | 110111 | 101000 | 100111 | 011000 | 010111 |
| 110 | 110001 | 111110 | 000001 | 001110 | 010001 | 011110 | 100001 | 101110 |
| 111 | 001111 | 000000 | 111111 | 110000 | 101111 | 100000 | 011111 | 010000 |

Figure 17: Order 8 magic square in $F_2{}^4$.

Similarly to what we did previously in Figure 8 and Figure 9, we first consider each digit of our magic square in $F_2$. The $i^{th}$ digit of each entry will correspond to $V_i$, hence giving us the tables for 6 value functions of each $V_i$. These functions of $V_i$ are just affine linear functions amongst each other as seen in Figure 19. First we define a natural basis of $F_2{}^6$, namely ($e_1$, $e_2$, $e_3$, $e_4$, $e_5$, $e_6$) as tabulated in Figure 19, and its corresponding dual basis in $F_2$, particularily ($f_1$, $f_2$, $f_3$, $f_4$, $f_5$, $f_6$). Similarly to Figure 4, we now have a basis table for $F_2{}^6$ and each linear function $f_i$ in $F_2$. So for example, taking the sum of the tables $f_3$, $f_4$, and $f_5$ will produce a table in $F_2$ that corresponds to $V_1$, the $1^{st}$ digit of each entry in our order 8 magic square in $F_2{}^6$.

| | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | 1 |
|---|---|---|---|---|---|---|---|
| $V_1$ | 0 | 0 | 1 | 1 | 1 | 0 | 1 |
| $V_2$ | 0 | 0 | 1 | 0 | 1 | 0 | 1 |
| $V_3$ | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| $V_4$ | 1 | 1 | 1 | 0 | 0 | 1 | 0 |
| $V_5$ | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| $V_6$ | 1 | 1 | 0 | 0 | 0 | 1 | 1 |

Figure 18: Associated value functions.

| | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|---|---|---|---|---|---|---|---|---|
| 000 | 000000 | 000001 | 000010 | 000011 | 000100 | 000101 | 000110 | 000111 |
| 001 | 001000 | 001001 | 001010 | 001011 | 001100 | 001101 | 001110 | 001111 |
| 010 | 010000 | 010001 | 010010 | 010011 | 010100 | 010101 | 010110 | 010111 |
| 011 | 011000 | 011001 | 011010 | 011011 | 011100 | 011101 | 011110 | 011111 |
| 100 | 100000 | 100001 | 100010 | 100011 | 100100 | 100101 | 100110 | 100111 |
| 101 | 101000 | 101001 | 101010 | 101011 | 101100 | 101101 | 101110 | 101111 |
| 110 | 110000 | 110001 | 110010 | 110011 | 110100 | 110101 | 110110 | 110111 |
| 111 | 111000 | 111001 | 111010 | 111011 | 111100 | 111101 | 111110 | 111111 |

Figure 19: Basis of ${F_2}^6$.

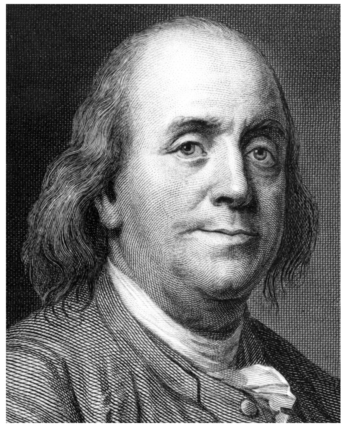
This order 8 magic square has a fascinating property in that several of its subspaces in ${F_2}^6$ preserve this magic property. In particular, of the 2-dimensional subspaces, we find ($e_3$, $e_6$), ($e_2 + e_3$, $e_6$), ($e_3$, $e_5 + e_6$), and ($e_2 + e_3$, $e_5 + e_6$).

For the 1st example, ($e_3$, $e_6$) represents the entries of Figure 19 spanned by the 3rd and 6th entries. So we turn our attention to the contiguous 2-by-2 blocks starting with the 1st one covering positions $000000_2$, $000001_2$, $001000_2$, $001001_2$ having value 52, 61, 14, and 3 respectively in our magic square. We can find the next contiguous 2-by-2 block by considering the affine parallel subspaces generated by adding $000010_2$ to each position, and so on. We notice that each of these 2-by-2 blocks independently share a sum of 130, half of the sum S for this magic square, 260.

For the next subspace, we consider the 2-by-2 broken-blocks with initial position $000000_2$, $000001_2$, $011000_2$, $011001_2$. This corresponds to the values 52, 61, 11, and 6 respectively in our magic square. Interestingly enough, these numbers also sum to 130, as well as the other affine subspaces parallel to this one.

The 3rd subspace says to focus on the 2-by-2 broken-blocks with initial positions $000000_2$, $000011_2$, $001000_2$, $001011_2$. This corresponds to values 42, 13, 14, and 51 respectively in our magic square. These subspaces add to 130, as well as the other affine parallel subspaces.

Once again, the fourth subspace says to focus on initial entries $000000_2$, $000011_2$, $011000_2$, $011011_2$. This corresponds to 52, 13, 11, and 54 respectively. Once again we see that their sum is 130.



Benjamin Franklin

Franklin made an interesting observation in that every "bent-row" of numbers shared the common sum of 260, which was the same sum S for this magic square. A bent-row descends diagonally from the 1st entry, then ascends diagonally from the 5th entry, wrapping around the square when necessary. To better understand this, he was refering to entries 52, 3, 5, 54, 43, 28, 30, 45 in his magic square as seen in Figure 15. What he found was the image of the vector subspace generated by the 3-dimensional basis ($e_3 + e_6$, $e_2 + e_5$, $e_2 + e_3 + e_4$). However, this only explains every 2nd bent row, only 4 of the 8 bent-rows. The other 4 bent-rows are the union of 2 affine parallel subspaces. So for example, the 2nd bent elbow beginning at entry 14 is the union of basis ($e_2 + e_3 + e_6$, $e_4 + e_5 + e_6$) which is the descending part, with ($e_1 + e_2 + e_3 + e_6$, $e_4 + e_5 + e_6$) which is the ascending part, to generate the entries 14, 60, 59, 10 and 23, 38, 37, 19 respectively.

# 6 Generating Magic Squares

First discovered by Robert Sedgewick and Kevin Wayne, when the order n of our magic square is odd, we may easily generate a magic square by first placing 1 in the bottom middle cell and repeatedly assign the next integer to the cell diagonally adjacent to the right and down. If the cell is already occupied, then we instead use the cell adjacently above, wrapping around the square when necessary. Figure 20 shows two examples of generated magic squares and their function calls. Following that is the provided source code, written in Java.

```
% java MagicSquare 3
 4  9  2
 3  5  7
 8  1  6

% java MagicSquare 5
11 18 25  2  9
10 12 19 21  3
 4  6 13 20 22
23  5  7 14 16
17 24  1  8 15
```

Figure 20: Example generations.

```java
public class MagicSquare {

    public static void main(String[] args) {
        int n = Integer.parseInt(args[0]);
        if (n % 2 == 0) throw new RuntimeException("n must be odd");

        // Instantiate our Square
        int[][] magic = new int[n][n];

        // Assign 1
        int row = n-1;
        int col = n/2;
        magic[row][col] = 1;

        // Fill Square
        for (int i = 2; i <= n*n; i++) {
            if (magic[(row + 1) % n][(col + 1) % n] == 0) {
                row = (row + 1) % n;
                col = (col + 1) % n;
            } else {
                // Don't Change Collumns
                row = (row - 1 + n) % n;
            }
            magic[row][col] = i;
        }

        // Print
        for (int i = 0; i < n; i++) {
            for (int j = 0; j < n; j++) {
                // Alignment
                if (magic[i][j] < 10)  System.out.print(" ");
                if (magic[i][j] < 100) System.out.print(" ");
                System.out.print(magic[i][j] + " ");
            }
            System.out.println();
        }
    }
}
```

# A Sources

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